Antibrackets and non-Abelian equivariant cohomology

A.Nersessian*

Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
Dubna, Moscow region, Russia 141980

Abstract

The Weil algebra of a semisimple Lie group and an exterior algebra of a symplectic manifold possess antibrackets. They are applied to formulate the models of non–abelian equivariant cohomologies.

*e-mail: nerses@thsun1.jinr.dubna.su

1 Introduction

The Batalin–Vilkovisky formalism (BV-formalism)[1] is the most adequate and powerful method for quantizing gauge fields; at the same time it is mathematically the most unusual formalism: its basic structure, antibracket (see Appendix A), is rather an exotic object.

Study of the BV geometry has shown that it is actually based on first principles of the theory of integration on supermanifolds, generalization of the Stokes theorem to pseudointegral (pseudodifferential) forms (see [2] and references therein), therefore the antibracket should play a fundamental role in (super)geometry.

Recently, in field theory, great interest has been displayed in equivariant cohomologies[1], mainly, in view of the application of the Duistermaat–Heckman localization formulae [3] connected w ith S^1-equivariant cohomology [4], to the calculation of path integrals of topological field theories (see [6] and references therein). In Ref. [7], the action of the $4d$ topological Yang–Mills theory was interpreted in terms of non–Abelian equivariant cohomology, whereas Atiah and Jeffrey interpreted its partition function as an Euler regularized equivariant class [8]. Witten proposed a generalization of the Duistermaat–Heckman formula to non–Abelian equivariant cohomologies and used it for calculating the partition function of the Yang–Mills $2d$ topological theory [9]. This work stimulated a more detailed study of non–Abelian equivariant cohomologies and search of related localization formulae [10].

Non–Abelian equivariant cohomologies are described by a number of equivariant models. Specifically, in Refs.[8], use was made of the Weil model having a natural geometric interpretation. The Cartan model convenient for treating the localization formulae was exploited in Refs.[9, 10]. The authors of Ref.[7] introduced the so–called BRST model adapted for field–theoretical problems. Kalkman put forward a parametric model of equivariant cohomology comprising the Weil and BRST models (the latter is naturally reduced to the Cartan model) [5].

In this note, we construct antibrackets on the basis of Weil algebra $W(g)$ of a Lie semisimple group G and the exterior algebra ΛM of a symplectic G-manifold (M, ω, G), with respect to which the operators of the exterior differentiation, contraction and Lie derivative, are (anti)Hamiltonian. Then we will be able to formulate the Weil and Cartan models for non–Abelian equivariant cohomology as well as a modified Kalkman model (further, the BRST and Cartan model) in an (anti)Hamiltonian manner.

The formulation suggested for non–Abelian equivariant cohomology is not only a convenient tool for describing equivariant cohomology, rather, it makes richer the theory of equivariant cohomology and topological field theories dressing them with the apparatus of BV–formalism and antisymplectic (super)geometry.

Note that the description of S^1-equivariant cohomology in terms of antibrackets [11] allowed one to connect thats with an interesting class of supersymmetric mechanics and to construct generalizations of S^1-equivariant characteristic classes.

[1] G–equivariant cohomology of G- manifold (M, G) is called the G–invariant cohomology of a quotient space M/G [4, 5].

2 Antibrackets on ΛM and $W(g)$.

Let us construct antibrackets on ΛM and $W(g)$ with respect to which the operators of differentiation, contraction with generators of the G-action and the Lie derivative are (anti)Hamiltonian.

Let (M, ω, G) be a compact symplectic G–invariant manifold; G be a semisimple Lie group; g be the Lie algebra of group G; $I_a(x)$, $a = 1, \dots \dim G$ be a generators of the symplectic G-action on (M, ω): $\omega^{-1}(dI_a, dI_b) = f^d_{ab} I_d$, where f^a_{bd} are structure constants of the Lie algebra g.

Let ΛM be an ($\mathbf{Z}$–graded) exterior algebra of (M, ω, G), and (x^i, θ^i) be its local coordinates (x^i are local coordinates of M; θ^i are the corresponding basic 1–forms: $\theta^i \leftrightarrow dx^i$). The $\mathbf{Z}$–grading on ΛM is given by the conditions $deg\ x^i = 0$, $deg\ \theta^i = 1$.

Then, the antibracket on ΛM is defined by the expression

$$\{f, g\}_\Lambda = \omega^{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial_l g}{\partial \theta_j} - \frac{\partial_r f}{\partial \theta_i} \frac{\partial g}{\partial x^j} \right) + \frac{\partial_r f}{\partial \theta^i} (\theta^k \partial \omega^{ij} / \partial x^k) \frac{\partial_l g}{\partial \theta_j} \tag{2.1}$$

where $\omega^{ij} \omega_{jk} = \delta^i_k$, $\omega_{ij} = \omega(\partial / \partial x^i, \partial / \partial x^j)$.

The functions

$$I_a(x), \quad Q_a = \frac{\partial I_a}{\partial x^i} \theta^i, \quad D_0 = -\frac{1}{2} \theta^i \omega_{ij} \theta^j, \tag{2.2}$$

define on ΛM the (anti)Hamiltonian vector fields corresponding, respectively, to the contraction of differential forms with the $\omega^{-1}(dI_a, \ )$, to the Lie derivative along $\omega^{-1}(dI_a, \ )$ and to the external differentiation [11].

Now consider the ($\mathbf{Z}$–graded) Weil algebra $W(g) = S(g^*) \otimes \Lambda(g^*)$ of the Lie group G. Here $S(g^*)$ is a symmetric algebra of polynomials on the algebra g^*, dual to g, with (commuting) coordinates ϕ^a; $\Lambda(g^*)$ is an external algebra on g^* with (anticommuting) coordinates c^a. The $\mathbf{Z}$–grading on $W(g)$ is given by the conditions $deg \phi^a = 2$, $deg c^a = 1$ (this grading is chosen due to correspondence of c^a and ϕ^a , respectively, to the the connection 1-form on the principal G–fiber bundle and to the its curvature).

The antibrackets on $W(g)$ can be defined by the formula

$$\{f, g\}_W = g^{ab} \left(\frac{\partial f}{\partial \phi^a} \frac{\partial_l g}{\partial c^b} - \frac{\partial_r f}{\partial c^a} \frac{\partial g}{\partial \phi^b} \right), \tag{2.3}$$

where $g^{ad} g_{db} = \delta^a_b$, and g_{ab} is the Cartan–Killing metric of algebra g.

The functions

$$\phi_a = g_{ab} \phi^b, \quad q_a = f^d_{ba} \phi_d c^b, \quad D = \frac{1}{2} g_{ab} \phi^a \phi^b - \frac{1}{2} f^d_{ab} \phi_d c^a c^b. \tag{2.4}$$

act on $W(g)$ as follows:

$$\{\phi_b, c^a\}_1 = \delta^a_b, \qquad \{\phi_b, \phi^a\}_1 = 0; \quad \{q_b, c^a\}_1 = f^a_{db} c^d, \qquad \{q_b, \phi^a\}_1 = f^a_{db} \phi^d. \tag{2.5}$$

$$\{D, c^a\}_1 = \phi^a - \tfrac{1}{2} f^a_{bd} c^b c^d \quad \{D, \phi^a\}_1 = -f^a_{bd} c^b \phi^d, \tag{2.6}$$

They define, respectively, the contraction of the connection 1–form with generators of the G-action on $W(g)$, co–adjoint action of G on $W(g)$ and the Weil differential.

Equipping the metric g_{ab} with the grading $deg\ \ g_{ab} = -2$, we obtain

$$deg\,\{\ \ ,\ \ \}_\alpha = -1; \quad deg\ \ (I^\alpha,\ \ Q^\alpha,\ \ D^\alpha) = (0,1,2), \tag{2.7}$$

where

$$\{\ \ ,\ \ \}_\alpha = (\{\ \ ,\ \ \}_\Lambda,\ \{\ \ ,\ \ \}_W), \quad (I^\alpha_a, Q^\alpha_a, D^\alpha) = ((I_a, Q_a, D_0), (\phi_a, q_a, D))\,. \tag{2.8}$$

The sets (2.2) and (2.4) form, with respect to a corresponding antibrackets, the superalgebra

$$\begin{array}{l} \{I^\alpha_a, Q^\alpha_b\}_\alpha = f^d_{ab} I^\alpha_d, \quad \{Q^\alpha_a, Q^\alpha_b\}_\alpha = f^d_{ab} Q^\alpha_d, \quad \{I^\alpha_a, D^\alpha\}_\alpha = Q^\alpha_a, \\ \{D^\alpha, D^\alpha\}_\alpha = 0, \quad \{Q^\alpha_a, D^\alpha\}_\alpha = 0, \quad \{I^\alpha_a, I^\alpha_b\}_\alpha = 0, \end{array} \tag{2.9}$$

3 Models for equivariant cohomology

Now, we are able to construct a models for equivariant cohomology.

The Weil model. On the space $\mathcal{A} = \Lambda(M) \otimes W(g)$, the antibracket

$$\{f, g\}_\mathcal{A} = \{f, g\}_\Lambda + \{f, g\}_W \tag{3.10}$$

is defined.

The functions

$$\mathcal{I}_a = (I_a + \phi_a), \quad \mathcal{Q}_a = (Q_a + q_a), \quad \mathcal{D} = (D_0 + D), \quad . \tag{3.11}$$

form on $(\mathcal{A}, \{\ \ ,\ \ \}_\mathcal{A})$ the superalgebra (2.10). They generate anti–Hamiltonian vector fields corresponding to the contraction with generators of the G-action , Lie derivative along them and to the total differential, respectively.

The G equivariant cohomology of the manifold M is defined as a subspace of $\mathcal{A}$ whose elements commute with all $\mathcal{I}_a$ and $\mathcal{Q}_a$ with respect to the antibrackets (3.10). It is a cohomology of the operator, generating by the function $\mathcal{D}$ [4, 5].

The Kalkman-like model. Let us deform the Weil model performing the (anti)canonical transformation preserving the $\mathbf{Z}$–grading and generated by the function

$$\Psi = -tc^a I_a(x), \ \ deg\ \ \Psi = 1, \tag{3.12}$$

where t is a parameter.

The antibracket (3.10) is invariant under these transformations by definition, and the Hamiltonians (3.11) are transformed to the following ones:

$$\begin{array}{rcl} \mathcal{I}^t_a & = & \mathcal{I}_a - t\{\mathcal{I}_a, \Psi\}_1 + \dfrac{t^2}{2!}\{\{\mathcal{I}_a, \Psi\}_1, \Psi\}_1 + ... = \\ & = & \phi_a + (1-t) I_a \qquad\qquad (3.13) \\ \mathcal{Q}^t_a & = & \mathcal{Q}_a - t\{\mathcal{Q}_a, \Psi\}_1 + \dfrac{t^2}{2!}\{\{\mathcal{Q}_a, \Psi\}_1, \Psi\}_1 + ... = \end{array}$$

$$= \mathcal{Q}_a. \tag{3.14}$$

$$\begin{aligned} \mathcal{D}^t &= \mathcal{D} - t\{\mathcal{D}, \Psi\}_1 + \frac{t^2}{2!}\{\{\mathcal{D}, \Psi\}_1, \Psi\}_1 + ... = \\ &= \mathcal{D} - t\phi^a I_a + \frac{t^2}{2} g^{ab} I_a I_b + tc^a Q_a + \frac{t(1-t)}{2} f^c_{ab} c^a c^b I_c \end{aligned} \tag{3.15}$$

They form the superalgebra (2.10) with respect to (3.10) at any values of the parameter t. The operators defined by the functions (3.13) and (3.14) coincide with the corresponding operators of the Kalkman parametric model for equivariant cohomology [5]. The equivariant cohomology in this model is defined like in the Weil model.

The differential given by the function (3.15) differs from that proposed by Kalkman:

$$\mathcal{D}^K = \{\mathcal{D},\ \}_\mathcal{A} - t(\phi^a - tI_a g^{ab})\{I_a,\ \}_\mathcal{A} + \frac{t(1-t)}{2} f^c_{ab} c^a c^b \{I_a,\ \}_\mathcal{A} - -tc^a\{Q_a,\ \}_\mathcal{A}, \tag{3.16}$$

that is not Hamiltonian with respect to (3.10).

BRST-like model. At $t = 1$ the functions (3.13)–(3.15) assume the form

$$\mathcal{I}^1_a = \phi_a, \quad \mathcal{Q}^1_a = \mathcal{Q}_a, \quad \mathcal{D}^1 = \mathcal{D} - \phi^a I_a + \frac{1}{2} g^{ab} I_a I_b + c^a Q_a. \tag{3.17}$$

Generators generated by the first two functions coincide with the corresponding generators of the standard BRST model (arising in the $4d$ topological Yang– Mills theory [7]) to which the Kalkman model is reduced at $t = 1$.

The function $\mathcal{D}^1$ generates the differential different from the standard BRST one

$$\hat{\mathcal{D}}^{BRST} = \{\mathcal{D},\ \}_\mathcal{A} - \phi^a\{I_a,\ \}_\mathcal{A} + c^a\{Q_a,\ \}_\mathcal{A}. \tag{3.18}$$

The equivariant cohomology in the BRST model belongs to the space $\Lambda(M) \otimes S(g^*)$ on which by the expression (2.1) the degenerate antibracket is defined.

Cartan-like models. Restriction of the BRST model to the space $\Lambda(M)\otimes S(g^*)$ results in the Cartan model. It is defined by the functions $(D_0 \pm I_\phi) \equiv D_0 \pm \phi^a I_a, \quad Q_\phi \equiv \phi^a Q_a$, forming the superalgebra

$$\begin{gathered} \{I_\phi \pm D_0, I_\phi \pm D_0\}_1 = \pm 2Q_\phi, \ \ \{I_\phi + D_0, I_\phi - D_0\}_1 = 0 \\ \{I_\phi \pm D_0, Q_\phi\}_1 = \{Q_\phi, Q_\phi\}_1 = 0. \end{gathered} \tag{3.19}$$

The function $D_0 - I_\phi$ determines the differential in the Cartan model nilpotent on the space of G–invariant elements of the space ΛM. Therefore, the G-equivariant cohomology of the manifold ΛM is an element of $S(g^*) \otimes \Lambda M$ commuting with $D_0 - I_\phi$ with respect to the antibracket (2.1) [5].

The restriction of the differential of the standard BRST model to $S(g^*) \otimes \Lambda M$ gives rise to that of the standard Cartan model. The limitation of the differential generated by the function (3.15) at $t = 1$ results in a different differential

$$\mathcal{D}^1_C = D_0 + \frac{1}{2} g^{ab}(\phi_a - I_a)(\phi_b - I_b): \ \ \{\mathcal{D}^1_C, \mathcal{D}^1_C\}_1 = (\phi^a - I^a)Q_a. \tag{3.20}$$

Acknowledgements. The author is thankful to A. Hietamäki for useful comments, and to I.A. Batalin and A.Niemi for interest in the work. This work was supported by the Grant No. M2I300 from International Scientific Foundation and by the INTAS Grant No. 93–2492) and has been carry out within the research program of the International Center of Fundamental Physics in Moscow.

Appendix

A Antibrackets

The antibracket of the functions $f(z)$, $g(z)$ on the supermanifold $\mathcal{M}$ is called the operation

$$\{f,g\} = \frac{\partial_r f}{\partial z^A}\Omega_1^{AB}\frac{\partial_l g}{\partial z^B}, \tag{A1}$$

(where r and l denote, respectively, the right– and left–handed derivatives) obeying the conditions

$$p(\{f,g\}) = p(f) + p(g) + 1, \quad \text{(gradingcondition)}$$
$$\{f,g\} = -(-1)^{(p(f)+1)(p(g)+1)}\{g,f\}, \quad (\text{“antisymmetrysity”}) \tag{A2}$$
$$\{f,\{g,h\}\} - (-1)^{(p(f)+1)(p(h)+1)}\{g,\{f,h\}\} = \{\{f,g\},h\}. \quad \text{(Jacobi id.)} \tag{A3}$$

With every function f, the antibracket associates an operator (anti–Hamiltonian vector field) of opposite parity $\hat{f} = \{f, \ \}$, and, in view of the Jacobi identity (A3), there holds the following relation:

$$\widehat{\{f,g\}} = \hat{f}\hat{g} - (-1)^{p(\hat{f})p(\hat{h})}\hat{g}\hat{f}.$$

These fields generate transformations preserving the antibracket (anticanonical transformations).

On the $(n.n)$–dimensional supermanifold, antibrackets can be nondegenerate. Then they can be associated with the antisymplectic structure

$$\Omega = dz^A\Omega_{AB}dz^B, \quad d\Omega = 0, \quad \Omega_{AB}\Omega_1^{BC} = \delta_A^C \tag{A4}$$

Locally, the antibrackets are reducible to the canonical form [12]

$$\Omega^{\rm can} = \sum_{i=1}^n dx^i \wedge d\eta_i, \quad \{f,g\}^{\rm can} = \sum_{i=1}^n \left(\frac{\partial_r f}{\partial x^i}\frac{\partial_l g}{\partial \eta_i} - \frac{\partial_r f}{\partial \eta_i}\frac{\partial_l g}{\partial x^i}\right), \tag{A5}$$

where $p(\eta_i) = p(x_i) + 1$. On the space $\Lambda_* M$ of polyvector fields of the manifold M, the canonical antibracket can be defined globally; in this case x^i are local coordinates of M and η_i are basis vector fields: $\eta_i \leftrightarrow \frac{\tilde{\partial}}{\partial x^i}$.

Antisymplectic structures on ΛM and $W(g)$ corresponding to the antibrackets (2.1) and (2.3) are, respectively, given by the expressions

$$\Omega_\Lambda = \omega_{ij}dx^i \wedge d\theta^j + \frac{1}{2}\omega_{ij,k}\theta^k dx^i \wedge dx^j, \quad \Omega_W = g_{ab}d\phi^a \wedge dc^b. \tag{A6}$$

References

[1] I. A. Batalin , G. A. Vilkovisky , Phys. Lett.**B102**; Phys. Rev. **D28** (1983) 2567 [E: **D30** (1984) 508]; (1981), 27; Nucl.Phys. **B234** (1984), 106

[2] A. Schwarz, Comm. Math.Phys. **155** (1993), 249;

H.Hata, B. Zwiebach, Ann. Phys (N.Y.) **229**(1994), 177

O. Khudaverdian, hep-th/9508174

[3] J.J. Duistermaat , G.J. Heckman, Inv. Math. **69** (1982) 259 ; *ibid***72** (1983), 153

[4] M. F. Atiah, R. Bott, Topology, **23**, No. 1 (1984) 1

V. Mathai, D. Quillen, Topology, **25** (1986) 85

[5] J. Kalkman, Comm. Math. Phys. **153** (1993) 447

[6] E. Keski- Vakkuri, A. J. Niemi, G. W. Semenoff, O. Tirkkonen, Phys. Rev. **D44** (1991), 3899

A. J. Niemi, O. Tirkkonen, Ann. Phys (N.Y),**235** (1994), 318

M. Blau, G. Thompson, J. Math. Phys **36** (1995), 2192

[7] S. Ouvry, R. Stora, P. van Baal, Phys. Lett. **B220** (1989) 159

[8] M. Atiah, L. Jeffrey, J. Geom. Phys. **7** (1991) 119

[9] E. Witten, J. Geom. Phys. **9** (1992) 303

[10] L.C. Jeffrey , F.C. Kirwan, Localization for non-Abelian group action., alg-geom/9307001

J. Kalkman, Residues in nonabelian localization., hep-th/9407115.

[11] A. Nersessian, JETP Lett., **58** No. 1 (1993), 66; NATO ASI Series **B331** (1994), 353; Preprint JINR E2-94-377 (1994)

[12] D. Leites, DAN SSSR, **236** (1977), 804 (in Russian)